Malaysian Journal of Mathematical Sciences 19(2): 383–398 (2025)
https://doi.org/10.47836/mjms.19.2.02

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Pseudo-Quasi Conformal Curvature Tensor of Quasi Sasakian Manifold with Generalized Sasakian Space Forms

AlHusseini, F. H.*[1] and Abood, H. M.[2]

[1]*Collage of Education, Department of Mathematics, University of Al-Qadisiyah, Al-Qadisiyah, Iraq*
[2]*Collage of Education for Pure Sciences, Department of Mathematics, University of Basrah, Basrah, Iraq*

E-mail: habeeb.abood@uobasrah.edu.iq
**Corresponding author*

Received: 12 May 2024
Accepted: 4 November 2024

Abstract

The current study focuses on three main topics; the pseudo-quasi conformal curvature tensor, quasi-Sasakian manifolds ($\mathcal{QSA}_S$-manifold), and generalized Sasakian space forms ($\mathcal{GS}$-space forms) employing the G-adjoined structure space. For a $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms, the components of the Ricci tensor and pseudo-quasi conformal curvature tensor are computed. Various types of $\mathcal{QSA}_S$-manifold are described, and their interactions with $\mathcal{GS}$-space forms are investigated. It has been shown that ξ-pseudo quasi conformally flat $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms includes quasi Einstein manifold. Furthermore, the condition of a quasi-pseudo quasi conformal $\mathcal{QSA}_S$-manifold of space forms to be a quasi-Einstein manifold is identified. Finally, the scalar curvature is determined for ξ-pseudo quasi-conformally flat $\mathcal{QSA}_S$-manifold and quasi-pseudo quasi conformal $\mathcal{QSA}_S$-manifold.

Keywords: quasi Sasakian manifold; generalized Sasakian space forms; pseudo-quasi conformal curvature tensor; quasi Einstein manifold.

1 Introduction

Normal manifolds are an important class of contact metric manifolds. However, the curvature character of such manifolds is unfamiliar in general, with the exception of Sasakian or cosymplectic manifolds. If the almost contact structure is normal and the fundamental 2-form is closed, the manifold $\mathcal{M}$ is referred to as quasi-Sasakian manifold.

Blair [6] proposed the concept of a quasi-Sasakian ($\mathcal{QSA}_S$) manifold as an integrated structure that includes both cosymplectic and Sasakian structures. Blair's demonstration of important geometric properties and the first instances of $\mathcal{QSA}_S$-manifolds make this generalization a valuable source for differential geometric research. Many authors have made significant contributions to our understanding of $\mathcal{QSA}_S$-manifolds. Kanemaki [11] established the necessary and sufficient condition for an almost contact metric manifold to be quasi-Sasakian. In a subsequent study, Kanemaki [12] determined the necessary and sufficient conditions for a quasi-Sasakian structure to be Sasakian and cosymplectic. Tanno [23] investigated the quasi-Sasakian manifold of odd rank. Olszak [18] identified the necessary and sufficient condition for a quasi-Sasakian manifold to be conformally flat, whereas Kirichenko and Rustanov [15] characterized the $\mathcal{QSA}_S$-Einstein manifold and showed the symmetry properties of its Riemannian curvature tensor, identifying new subclasses of $\mathcal{QSA}_S$-manifold.

The study of submanifolds within $\mathcal{QSA}_S$-manifolds has also received significant attention. De et al. [10] demonstrated that if the typical submanifolds of $\mathcal{QSA}_S$-manifolds are $T(M)$-invariant in the context of a non-zero tensor field F, then the corresponding distribution D is not integrable. Mondal and De [17] demonstrated locally ϕ-Ricci symmetric and ϕ-Ricci symmetric of three-dimensional $\mathcal{QSA}_S$-manifolds with constant structure functions, as well as cyclic parallel and η-parallel Ricci tensor. Perktaş and Yildis [19] investigated η-Ricci solitons, gradient Ricci solitons, Ricci solitons and Yamabe solitons in three-dimensional $\mathcal{QSA}_S$-manifold employing the Schouten-van-Kampen connection, presenting various examples of these manifolds. Rahman [20] investigated warped product submanifolds of $\mathcal{QSA}_S$-manifold, specifically concentrating on warped product CR-submanifolds.

On the related side, the geometric properties and applications of $\mathcal{GS}$-space forms have been thoroughly examined. Carriazo [7] provided examples of $\mathcal{GS}$-space forms employing the conformal changes of metric and warped product. Alegre and Carriazo [4] demonstrated that constant functions required for α-Sasakian of $\mathcal{GS}$-space forms of dimensions five or higher. On the other hand, Venkatesha and Shanmukha [24] examined W_2-locally symmetric, W_2-pseudo-symmetric, W_2-ϕ-recurrent, and W_2-locally ϕ-symmetric $\mathcal{GS}$-space forms. Venkatesha et al. [25] showed that locally ϕ-recurrent $\mathcal{GS}$-space forms are manifolds of constant curvature. Vidyavathi et al. [26] investigated the semi-symmetric characteristics of $\mathcal{GS}$-space forms. On the other hand, Dong [16] considered the real hypersurface in complex space form, which implies that this manifold has an almost contact metric structure.

Many earlier studies have concentrated on some of the curvature tensors, including the Riemannian curvature tensor, which is regarded as a key component of the pseudo-quasi conformal curvature tensor under consideration in this study. Abood and Al-Hussaini [2] demonstrated the geometric properties of conharmonic curvature tensor of locally conformal almost cosymplectic manifold. Abu-Saleem and Rustanov [3] examined several curvature identities provided by the Riemannian curvature tensor. Al-Hussaini et al. [5] investigated the vanishing conharmonic curvature tensor of normal locally conformal almost cosymplectic manifold.

Although substantial progress has been created, there are still some fundamental questions, particularly about the relation between $\mathcal{QSA}_S$-manifolds and $\mathcal{GS}$-space forms, as well as their general geometric relevance. Studying these manifolds not only improves our comprehension of differential geometry, but also helps us understand more general mathematical and physical theories like curvature, symmetry, and Ricci solitons. This article seeks to fill gaps in the literature by addressing pressing concerns and providing new information on the structure and properties of $\mathcal{GS}$-space forms and $\mathcal{QSA}_S$-manifolds.

2 Preliminaries

In this section, numerous ideas and information about the topic of this article are discussed. The structure equations and components of the Riemannian curvature tensor for $\mathcal{QSA}_S$-manifolds are derived.

Definition 2.1. [6] *An almost contact metric manifold ($\mathcal{ACOM}$-manifold) is a smooth manifold $\mathcal{M}$ equipped with a quadruple $(\eta, \xi, \varphi, \mathcal{G})$ where,*

- η *is a contact form,*
- ξ *is a characteristic vector,*
- Φ *is endomorphisim tensor of type $(1;1)$,*
- $\mathcal{G} = \langle .,. \rangle$ *is a Riemannian metric.*

Moreover, the following conditions are satisfied,

1. $\eta(\xi) = 1$,
2. $\Phi(\xi) = 0$,
3. $\eta \circ \Phi = 0$,
4. $\Phi^2 = -id + \eta \otimes \xi$,
5. $\mathcal{G}(\Phi W, \Phi P) = \mathcal{G}(W, P) - \eta(W)\eta(P),\ W, P \in \mathfrak{X}(\mathcal{M})$.

In this paper, we employ the $\mathcal{AG}_S$-space to more precisely characterize structural equations. Kirichenko [13] provide additional information and specifics on the building of the adjoined G-structure ($\mathcal{AG}_S$-space).

Lemma 2.1. [13] *In the $\mathcal{AG}_S$-space, the matrices representing the tensors $\mathcal{G}$ and Φ are defined as follows:*

$$\mathcal{G}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{bmatrix}, \quad \Phi^i_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & 0 \\ 0 & 0 & -\sqrt{-1}I_n \end{bmatrix},$$

where I_n is the identity matrix.

Definition 2.2. [6] *A quasi-Sasakian Structure ($\mathcal{QSA}_S$-structure) is an $\mathcal{ACOM}$-structure characterized by a closed fundamental form $\Omega(W, P) = \mathcal{G}(W, \Phi P)$ and the condition $2\, d\eta \otimes \xi + \mathcal{N}_\Phi = 0$, where $\mathcal{N}_\Phi$ is a $(2, 1)$-tensor known as the Nijenhuis tensor of Φ [21]. A manifold $\mathcal{M}$ equipped with a $\mathcal{QSA}_S$-structure is called a $\mathcal{QSA}_S$-manifold.*

Lemma 2.2. [15] *By adopting the $\mathcal{AG}_S$-space, the complete structure equations of the $\mathcal{QSA}_S$-structure are provided below,*

1. $d\omega^a = \omega^a_b \wedge \omega^b + B^a_b \omega \wedge \omega^b$,
2. $d\omega_a = -\omega^b_a \wedge \omega_b - B^b_a \omega \wedge \omega_b$,
3. $d\omega = 2B^a_b \omega^b \wedge \omega_a$,
4. $d\omega^a_b - \omega^a_c \wedge \omega^c_b = (A^{ad}_{bc} - 2B^a_b B^d_c)\omega^c \wedge \omega_d + B^a_{bc}\omega \wedge \omega^c + B^{ac}_b \omega \wedge \omega_c$,
5. $dB^a_b + B^e_b \omega^e_a - B^a_e \omega^e_b = B^{bc}_a \omega^c + B^{ac}_b \omega_c$,

where

1. $B^{[bc]}_a = B^a_{[bc]} = 0, \overline{B^a_{bc}} = -B^{bc}_a$,
2. $\overline{A^{ad}_{bc}} = A^{bc}_{ad}; A^{ad}_{[bc]} = A^{[ad]}_{bc} = 0.$

The notations [ ] and $\overline{A^{ad}_{bc}}$ above refer to the alternative indices and the conjugate operator, respectively.

Lemma 2.3. [15] *By employing the $\mathcal{AG}_S$-space, the components of the Riemann-Christoffel tensor of the $\mathcal{QSA}_S$-manifold are presented below,*

1. $\mathcal{R}_{\hat{a}\hat{b}cd} = B^a_{[c} B^d_{b]}$,
2. $\mathcal{R}_{\hat{a}b0c} = B^a_{bc}$,
3. $\mathcal{R}_{\hat{a}0b0} = B^a_c B^c_b$,
4. $\mathcal{R}_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - B^a_c B^d_b - 2B^a_b B^d_c$,
5. $\mathcal{R}_{\hat{a}b0\hat{c}} = B^{ac}_b$.

The Ricci tensor is a $(2,0)$-tensor defined as $\mathcal{S}_{th} = -\mathcal{R}^i_{thi}$ [8]. In the $\mathcal{AG}_S$-space for a $\mathcal{QSA}_S$-manifold [15], the components of the Ricci tensor are expressed as follows,

$$\begin{aligned} \mathcal{S}_{\hat{a}b} &= \mathcal{S}_{b\hat{a}} = 2B^a_c B^c_b - A^{ac}_{bc}, \\ \mathcal{S}_{a0} &= \mathcal{S}_{0a} = B^c_{ac}, \\ \mathcal{S}_{\hat{a}0} &= \mathcal{S}_{0\hat{a}} = -B^{ac}_c, \\ \mathcal{S}_{00} &= -2B^a_c B^c_a. \end{aligned} \tag{1}$$

Otherwise, the components are zero.

On the other hand, the scalar curvature of a $\mathcal{QSA}_S$-manifold, using the $\mathcal{AG}_S$-space is calculated based on the following formula,

$$\mathcal{K} = \mathcal{G}^{th}\mathcal{S}_{th} = 2(B^a_c B^c_a - A^{ac}_{ac}).$$

Definition 2.3. [9] *An $\mathcal{ACOM}$-manifold satisfying the condition,*

$$\mathcal{S}(W,P) = \alpha\mathcal{G}(W,P) + \beta\eta(W)\eta(P) \; \forall \; W,P \in \mathfrak{X}(\mathcal{M}),$$

is called a quasi-Einstein manifold. In a special case where β vanishes, the manifold reduces to an Einstein manifold. Here, α and β are smooth maps.

Definition 2.4. [2] *In an $\mathcal{ACOM}$-manifold, the condition for the Φ-invariant property is given by,*

$$\Phi \circ \mathcal{S} = \mathcal{S} \circ \Phi.$$

In the $\mathcal{AG}_S$-space, this condition takes the following form,

$$\mathcal{S}^{\hat{a}}_{b} = \mathcal{S}^{\hat{a}}_{0} = 0.$$

Definition 2.5. [14] *An $\mathcal{ACOM}$-manifold is said to have a point constant Φ-holomorphic sectional curvature (or a point constant $\Phi\mathcal{HS}$-curvature for short) if,*

$$\langle \mathcal{R}(W, \Phi W, W, \Phi W,)\rangle = h\|W\|^4,$$

where $h \in C^{\infty}(\mathcal{M})$ is a smooth function, and $W \in \mathfrak{X}(\mathcal{M})$ is a vector field on $\mathcal{M}$.

Theorem 2.1. *A $\mathcal{QSA}_S$-manifold has a point constant $\Phi\mathcal{HS}$-curvature h iff, on the $\mathcal{AG}_S$-space, the condition below holds,*

$$A^{ad}_{bc} = -\frac{h}{2}\tilde{\delta}^{ad}_{bc} + 3B^{(a}_{(b}B^{d)}_{c)},$$

where the brackets ( ) indicate the symmetric indices and $\tilde{\delta}^{ad}_{bc}$ is the Kronecker delta of the second type, given by the formula $\tilde{\delta}^{ad}_{bc} = \delta^a_b\delta^d_c + \delta^a_c\delta^d_b$.

Definition 2.6. *A $\mathcal{QSA}_S$-manifold is said to have a point constant $\Phi\mathcal{HQS}$-curvature if,*

$$\langle \mathcal{Q}(W, \Phi W, W, \Phi W)\rangle = h\|W\|^4,$$

where $h \in C^{\infty}(\mathcal{M})$ for all $W \in \mathfrak{X}(\mathcal{M})$.

Definition 2.7. [22] *Let $(\mathcal{M}, \eta, \xi, \Phi, \mathcal{G})$ be an $\mathcal{ACOM}$-manifold of dimension $2n + 1$. A pseudo quasi-conformal curvature tensor is a $(4, 0)$-tensor $\mathcal{Q}$ defined as,*

$$\begin{aligned}\mathcal{Q}_{ijkl} =& (\mathrm{p} + \mathrm{r})\mathcal{R}_{ijkl} + (\mathrm{q} - \frac{\mathrm{r}}{2n})(\mathcal{S}_{jl}\mathcal{G}_{ik} - \mathcal{S}_{il}\mathcal{G}_{jk}) + \mathrm{q}(\mathcal{S}_{ik}\mathcal{G}_{jl} - \mathcal{S}_{jk}\mathcal{G}_{il}) \\ &- \frac{\mathcal{K}}{2n(2n+1)}(\mathrm{p} + 4n\mathrm{q})[\mathcal{G}_{jl}\mathcal{G}_{ik} - \mathcal{G}_{il}\mathcal{G}_{jk}],\end{aligned}$$

and satisfies the following properties,

$$\mathcal{Q}_{ijkl} = -\mathcal{Q}_{jikl} = -\mathcal{Q}_{ijlk} = \mathcal{Q}_{klij}, \mathcal{Q}_{ijkl} = \mathcal{Q}_{iklj} + \mathcal{Q}_{iljk} = 0,$$

where p, r and q are constants and that $\mathrm{p}^2 + \mathrm{q}^2 + \mathrm{r}^2 > 0$.

3 $\mathcal{GS}$-Space Forms and $\Phi\mathcal{HQS}$-Curvature

This section discusses the pseudo-quasi conformal curvature tensor and its role in classifying $\mathcal{QSA}_S$-manifolds as $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$. It demonstrates the essential connections between $\mathcal{QSA}_S$ manifolds and $\mathcal{GS}$-space forms required for understanding curvature behavior and identifying Einstein or quasi-Einstein singularities.

The forthcoming theorem computes the components of the pseudo-quasi conformal curvature tensor for a $\mathcal{QSA}_S$-manifold, laying the groundwork for their geometric analysis. We find the curvature components of the tensor $\mathcal{Q}$ by directly applying Definition 2.7 and employing the $\mathcal{AG}_S$-space.

Theorem 3.1. *Using the $\mathcal{AG}_S$-space, the components of the pseudo-quasi conformal curvature tensor of a $\mathcal{QSA}_S$-manifold are given as follows,*

1. $\mathcal{Q}_{\hat{a}bc\hat{d}} = (\mathrm{p}+\mathrm{r})(A^{ad}_{bc} - 2B^a_b B^d_c - B^a_c B^d_b) + \left(\mathrm{q} - \frac{\mathrm{r}}{2n}\right)\mathcal{S}^d_b\delta^a_c + \mathrm{q}\mathcal{S}^a_c\delta^d_b - \frac{\mathcal{K}\delta^d_b\delta^a_c}{2n(2n+1)}(\mathrm{p}+4n\mathrm{q}),$
2. $\mathcal{Q}_{\hat{a}b0c} = (\mathrm{p}+\mathrm{r})B^a_{bc} + \mathrm{q}\mathcal{S}_{b0}\delta^a_c,$
3. $\mathcal{Q}_{\hat{a}b0\hat{c}} = (\mathrm{p}+\mathrm{r})B^{ac}_b + \mathrm{q}\mathcal{S}^a_0\delta^c_b,$
4. $\mathcal{Q}_{\hat{a}0b0} = (\mathrm{p}+\mathrm{r})B^a_c B^c_b + \left(\mathrm{q} - \frac{\mathrm{r}}{2n}\right)\mathcal{S}_{00}\delta^a_b + \mathrm{q}\mathcal{S}^a_b g_{00} - \frac{\mathcal{K}\delta^a_b}{2n(2n+1)}(\mathrm{p}+4n\mathrm{q}),$
5. $\mathcal{Q}_{\hat{a}\hat{b}cd} = 2(\mathrm{p}+\mathrm{r})B^a_{[c}B^b_{d]} + \left(\frac{\mathrm{r}}{n}\right)\mathcal{S}^{[a}_d\delta^{b]}_c + 4\mathrm{q}\mathcal{S}^{[a}_{[c}\delta^{b]}_{d]} - \frac{\mathcal{K}\delta^{ab}_{cd}}{2n(2n+1)}(\mathrm{p}+4n\mathrm{q}).$

The upcoming theorem refines Theorem 3.1 by defining the criteria for $\mathcal{QSA}_S$-manifolds to have a point constant $\Phi\mathcal{HQS}$-curvature.

Theorem 3.2. *A $\mathcal{QSA}_S$-manifold has point constant $\Phi\mathcal{HQS}$-curvature h iff, on the $\mathcal{AG}_S$-space, the following condition is satisfied,*

$$A^{ad}_{bc} = 3B^{(a}_{(b}B^{d)}_{c)} - \left(2\mathrm{q} - \frac{\mathrm{r}}{2n}\right)\mathcal{S}^{(a}_{(d}\delta^{c)}_{b)} - \frac{1}{(\mathrm{p}+\mathrm{r})}\left[\frac{h}{2} - \frac{(\mathrm{p}+4n\mathrm{q})\mathcal{K}}{2n(2n+1)}\right]\tilde{\delta}^{ad}_{bc}.$$

Proof. Assume that $\mathcal{M}$ is a $\mathcal{QSA}_S$-manifold with point constant $\Phi\mathcal{HQS}$-curvature. In accordance with Definition 2.6, we have,

$$\langle \mathcal{Q}(W, \Phi W, W, \Phi W,)\rangle = h\|W\|^4.$$

In the $\mathcal{AG}_S$-space, it follows that,

$$\mathcal{Q}_{ijkl}W^i(\Phi W)^j W^k(\Phi W)^l = h\mathcal{G}_{ij}\mathcal{G}_{kl}W^iW^jW^kW^l.$$

By employing the properties $(\Phi W)^a = \sqrt{-1}W^a$, $(\Phi W)^{\hat{a}} = -\sqrt{-1}W^{\hat{a}}$, $(\Phi W)^0 = 0$ as well as the properties of the pseudo-quasi conformal tensor, we deduce,

$$4\mathcal{Q}_{\hat{a}bc\hat{d}} = 4h\delta^a_b\delta^d_c.$$

Thus, the necessary and sufficient condition for an $\mathcal{ACOM}$-manifold to have point-constant $\Phi\mathcal{HQS}$-curvature is,

$$\mathcal{Q}^{(a}{}_{(bc)}{}^{d)} = \frac{h}{2}\tilde{\delta}^{ad}_{bc}. \quad (2)$$

Let $\mathcal{M}$ be a $\mathcal{QSA}_S$-manifold with point constant $\Phi\mathcal{HQS}$-curvature tensor, we have,

$$\mathcal{Q}^a{}_{bc}{}^d = (\mathrm{p}+\mathrm{r})(A^{ad}_{bc} - 2B^a_b B^d_c - B^a_c B^d_b) + \left(\mathrm{q} - \frac{\mathrm{r}}{2n}\right)\mathcal{S}^d_b\delta^a_c + \mathrm{q}\mathcal{S}^a_c\delta^d_b - \frac{\mathcal{K}\delta^d_b\delta^a_c}{2n(2n+1)}(\mathrm{p}+4n\mathrm{q}). \quad (3)$$

Symmetrizing (3) with respect to the indices (b, c) and (a, d) and then using (2), we obtain,

$$A^{ad}_{bc} = 3B^{(a}_{(b}B^{d)}_{c)} - \left(2\mathrm{q} - \frac{\mathrm{r}}{2n}\right)\mathcal{S}^{(a}_{(d}\delta^{c)}_{b)} - \frac{1}{(\mathrm{p}+\mathrm{r})}\left[\frac{h}{2} - \frac{(\mathrm{p}+4n\mathrm{q})\mathcal{K}}{2n(2n+1)}\right]\tilde{\delta}^{ad}_{bc}.$$

Conversely, we can deduce this condition directly by substituting the holomorphic sectional curvature tensor into $\mathcal{Q}^{(a}{}_{(bc)}{}^{d)}$. □

Theorem 3.3 establishes requirements for a $\mathcal{QSA}_S$-Einstein manifold by linking the proportionality of the Ricci tensor to the metric tensor.

Theorem 3.3. *A $\mathcal{QSA}_S$-manifold is an Einstein manifold iff, the following conditions hold,*

$$\alpha = -2B^a_c B^c_a, \quad B^c_{ac} = 0, \quad A^{ac}_{bc} = 2B^a_c B^c_b + 2B^a_c B^c_a.$$

Proof. Assume that $\mathcal{M}$ is an Einstein manifold, regarding to Definition 2.3, we have,

$$\mathcal{S}(W, P) = \alpha \mathcal{G}(W, P).$$

In local coordinates, this relation becomes $\mathcal{S}_{ij} = \alpha \mathcal{G}_{ij}$. Using the $\mathcal{AG}_S$-space, we deduce that $\mathcal{S}_{00} = \alpha \mathcal{G}_{00}$. By applying the relations in (1), we find that $\alpha = -2B^a_c B^c_a$, and from Lemma 2.1 we obtain $B^c_{ac} = 0$.

Now, $\mathcal{S}_{\hat{a}b} = \alpha \delta^a_b$ is equivalent to,

$$2B^a_c B^c_b - A^{ac}_{bc} = \alpha \delta^a_b.$$

Thus,

$$A^{ac}_{bc} = 2B^a_c B^c_b + 2B^a_c B^c_a.$$

The reverse implication of the theorem follows by employing the relations in (1) along with the conditions above. □

Corollary 3.1. *A $\mathcal{QSA}_S$-manifold is an Einstein manifold iff, it has Φ-invariant Ricci tensor and the conditions below hold,*

$$\alpha = -2B^a_c B^c_a, \quad A^{ac}_{bc} = 2B^a_c B^c_b + 2B^a_c B^c_a.$$

Proof. The desired result follows directly by applying the above theorem and Definition 2.4. □

The following theorem introduces quasi-Einstein $\mathcal{QSA}_S$-manifold, extending Theorem 3.3 with variable Ricci tensor proportionality.

Theorem 3.4. *A $\mathcal{QSA}_S$-manifold is a quasi-Einstein manifold iff, it satisfies the following conditions,*

$$\alpha + \beta = -2B^a_c B^c_a, \quad B^c_{ac} = 0, \quad A^{ac}_{bc} = 2B^a_c B^c_b + (2B^a_c B^c_a + \beta)\delta^a_b.$$

Proof. Assume that $\mathcal{M}$ is a quasi-Einstein manifold. According to Definition 2.3, we have $\mathcal{S}_{00} = \alpha + \beta$. Using the relations in (1) we find that $\alpha + \beta = -2B^a_c B^c_a$, and from Lemma 2.1, we obtain $B^c_{ac} = 0$.

Furthermore, the equality $\mathcal{S}_{\hat{a}b} = \alpha \mathcal{G}_{\hat{a}b}$ is equivalent to the equation,

$$A^{ac}_{bc} = 2B^a_c B^c_b + (2B^a_c B^c_a + \beta)\delta^a_b.$$

The reverse of the theorem also holds by employing the relations in (1) and the above conditions. □

Corollary 3.2. *A $\mathcal{QSA}_S$-manifold is an quasi-Einstein manifold iff, it has Φ-invariant Ricci tensor and the conditions below hold,*

$$\alpha + \beta = -2B^a_c B^c_a, \quad A^{ac}_{bc} = 2B^a_c B^c_b + (2B^a_c B^c_a + \beta)\delta^a_b.$$

Proof. This result follows directly by applying the above theorem and Definition 2.4. □

The following argument relates Einstein $\mathcal{QSA}_S$-manifolds to point constant $\Phi\mathcal{HS}$-curvature, expanding Theorem 3.3.

Theorem 3.5. *If a $\mathcal{QSA}_S$-manifold is an Einstein manifold with point constant $\Phi\mathcal{HS}$-curvature h, then $h = 0$.*

Proof. Assume that $\mathcal{M}$ is an Einstein $\mathcal{QSA}_S$-manifold, by Theorem 3.3, we have,

$$A^{ac}_{bc} = 2B^a_c B^c_b + 2B^a_c B^c_a, \tag{4}$$

Given that $\mathcal{M}$ has point constant $\Phi\mathcal{HS}$-curvature h, it follows that,

$$A^{ad}_{bc} = -\frac{(n+1)h}{2}\delta^a_b + 3B^{(a}_{(b}B^{d)}_{c)}, \tag{5}$$

Combining (4) and (5) and noting that $n > 1$, we deduce the required result. □

Definition 3.1. *An $\mathcal{QSA}_S$-manifold is classified as follows,*

1. $\mathcal{G}_1$ *if* $\mathcal{R}_{\hat{a}bcd} = 0$,
2. $\mathcal{G}_2$ *if* $\mathcal{R}_{\hat{a}\hat{b}cd} = 0$,
3. $\mathcal{G}_3$ *if* $\mathcal{R}_{\hat{a}bc\hat{d}} = 0$,
4. $\mathcal{G}_4$ *if* $\mathcal{R}_{a0b0} = \mathcal{R}_{\hat{a}0b0} = \mathcal{R}_{a0bc} = \mathcal{R}_{\hat{a}0bc} = \mathcal{R}_{a0\hat{b}c} = 0$.

The consequent Theorems 3.6–3.9 classify $\mathcal{QSA}_S$-manifolds into $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ categories according to curvature vanishing features and identities.

Theorem 3.6. *If a $\mathcal{QSA}_S$-manifold is of class $\mathcal{G}_4$, then its Riemannian curvature tensor satisfies the first special property,*

$$\eta \circ \mathcal{R}(\Phi^2 W, \Phi^2 P)\Phi^2 Z = \eta \circ (\mathcal{R}(\Phi W, \Phi P)\Phi^2 Z + \mathcal{R}(\Phi^2 W, \Phi P)\Phi Z + \mathcal{R}(\Phi W, \Phi^2 P)\Phi Z).$$

Theorem 3.7. *If a $\mathcal{QSA}_S$-manifold is of class $\mathcal{G}_4$, then its Riemannian curvature tensor satisfies the second special property,*

$$\eta \circ [\mathcal{R}(\Phi^2 W, \xi)\Phi^2 P + \eta \circ (\mathcal{R}(\Phi W, \xi)\Phi P] = 0.$$

Theorem 3.8. *Every $\mathcal{QSA}_S$-manifold is of class $\mathcal{G}_1$.*

Theorem 3.9. *Suppose $\mathcal{M}$ is a $\mathcal{QSA}_S$-manifold, then,*

1. *$\mathcal{M}$ is of class $\mathcal{G}_2$ iff,* $B^a_{[c}B^d_{b]} = 0$,
2. *$\mathcal{M}$ is of class $\mathcal{G}_3$ iff,* $A^{ad}_{bc} = B^a_c B^d_b + 2B^a_b B^d_c$,
3. *$\mathcal{M}$ is of class$\mathcal{G}_4$ iff,* $B^a_b = B^{ac}_b = 0$.

Proof. The desired results are derived from Definition 3.1 and Lemma 2.3. □

Definition 3.2. *A generalized Sasakian space forms ($\mathcal{GS}$-space forms) is an $\mathcal{ACOM}$-manifold, where there exist functions g_1, g_2 and g_3 on the manifold $\mathcal{M}$ such that,*

$$\begin{aligned}\mathcal{R}_{ijkl} =& g_1(\mathcal{G}_{ik}\mathcal{G}_{jl} - \mathcal{G}_{il}\mathcal{G}_{jk}) + g_2(\Omega_{il}\Omega_{kj} + 2\Omega_{ij}\Omega_{kl} - \Omega_{lj}\Omega_{ik}) \\ &+ g_3(\eta_j\eta_k\mathcal{G}_{il} - \eta_j\eta_l\mathcal{G}_{ik} + \eta_i\eta_l\mathcal{G}_{jk} - \eta_i\eta_k\mathcal{G}_{jl}). \end{aligned} \quad (6)$$

The subsequent Theorems 3.10–3.13 provide $\mathcal{GS}$-space forms for $\mathcal{QSA}_S$-manifolds and link functions g_1, g_2 and g_3 to their quasi-Einstein counterparts.

Theorem 3.10. [1] *A $\mathcal{GS}$-space forms is a quasi-Einstein manifold with $\alpha = 2ng_1 + 3g_2 - g_3$ and $\beta = -(3g_2 + (2n-1)g_3)$.*

Theorem 3.11. *On the $\mathcal{AG}_S$-space, a $\mathcal{QSA}_S$-manifold is a $\mathcal{GS}$-space forms iff,*

1. $B^a_c B^c_b = (g_1 - g_3)\delta^a_b$,
2. $A^{ad}_{bc} = (g_1 + g_2)\delta^a_c\delta^d_b + 2g_2\delta^a_b\delta^d_c + B^a_c B^d_b + 2B^a_b B^d_c$,
3. $2B^a_{[c}B^d_{b]} = (g_1 - g_2)\delta^a_c\delta^d_b + (g_2 + g_1)\delta^a_d\delta^b_c$,
4. $B^a_b = B^{ac}_b = 0$.

Proof. For $i = \hat{a},\ j = 0,\ k = b$ and $l = 0$, we have,

$$\begin{aligned}\mathcal{R}_{\hat{a}0b0} =& g_1(\mathcal{G}_{\hat{a}b}\mathcal{G}_{00} - \mathcal{G}_{\hat{a}0}\mathcal{G}_{0b}) + g_2(\Omega_{\hat{a}0}\Omega_{0b} + 2\Omega_{\hat{a}0}\Omega_{b0} - \Omega_{00}\Omega_{\hat{a}b}) \\ &+ g_3(\eta_0\eta_b\mathcal{G}_{\hat{a}0} - \eta_0\eta_0\mathcal{G}_{\hat{a}b} + \eta_{\hat{a}}\eta_0\mathcal{G}_{0b} - \eta_{\hat{a}}\eta_b\mathcal{G}_{00}),\end{aligned}$$

Using Theorem 2.3 and Lemma 2.1, we deduce,

$$B^a_c B^c_b = (g_1 - g_3)\delta^a_b.$$

The remaining requirements follow similarly. □

Theorem 3.12. *A $\mathcal{GS}$-space forms is a manifold of class,*

1. $\mathcal{G}_2$ *iff*, $g_1 = g_2$,
2. $\mathcal{G}_3$ *iff*, $g_1 = g_2 = 0$,
3. $\mathcal{G}_4$ *iff*, $g_1 = g_3$.

Proof. Using Theorem 3.11 and Definition 3.1, we conclude the required results. □

Theorem 3.13. *If a $\mathcal{QSA}_S$-manifold is a $\mathcal{GS}$-space forms with point constant $\Phi\mathcal{HS}$-curvature h, then,*

$$h = 0, \qquad g_1 = \frac{-3(n-1)}{8n^2 - 4n - 1}, \qquad g_2 = \frac{(n-1)}{8n^2 - 4n - 1}, \qquad g_3 = \frac{3n}{8n^2 - 4n - 1}.$$

Proof. From Theorems 2.1 and 3.11, we have,

$$(g_1 + g_2)\delta^a_c\delta^d_b + 2g_2\delta^a_b\delta^d_c + B^a_c B^d_b + 2B^a_b B^d_c = -\frac{h}{2}\tilde{\delta}^{ad}_{bc} + 3B^{(a}_{(b}B^{d)}_{c)}.$$

Taking the symmetry between indices (a, d) and (b, c) for the above equation, we deduce,

$$(g_1 + g_2)\tilde{\delta}^{ad}_{bc} + 2g_2\tilde{\delta}^{ad}_{bc} = -\frac{h}{2}\tilde{\delta}^{ad}_{bc}.$$

Consequently, since $\mathcal{GS}$-space forms are quasi-Einstein manifolds, we can deduce the value of h. Theorems 3.4 and 3.11 yield the values g_1, g_2, g_3. □

4 $\mathcal{GS}$-Space Forms for $\mathcal{QSA}_S$-Manifold

The present section applies the conclusions of Section 3 to $\mathcal{GS}$-space forms, defining the requirements for them to be Einstein or quasi-Einstein. It enhances classifications using ξ-$\mathcal{PQC}$-flat, Φ-$\mathcal{PQC}$-flat, and $\mathcal{QPQ}$-flat manifolds, connecting local curvature to global geometric structures. It builds on and explains the classifications and criteria introduced in Section 3, tying them to explicit scalar and tonsorial computations for advanced geometric analysis.

In the $\mathcal{AG}_S$-space, the component of Ricci tensor for the $\mathcal{QSA}_S$-manifold $\mathcal{GS}$-space forms are listed below,

$$\begin{aligned}
&\mathcal{S}_{\hat{a}b} = \mathcal{S}_{b\hat{a}} = -(g_1+g_2)\delta^a_b + 2ng_2\delta^a_b + 2B^a_cB^c_b,\\
&\mathcal{S}_{00} = -2n(g_1-g_3).
\end{aligned} \tag{7}$$

Otherwise, the components are zero.

The subsequent theorem demonstrates Ricci tensor invariance under Φ for $\mathcal{GS}$ forms, building on Theorems 3.3 and 3.10.

Theorem 4.1. *The $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms has a Φ-invariant Ricci tensor.*

The upcoming theorem specifies requirements for $\mathcal{GS}$-space forms to be Einstein, expanding Theorem 3.3.

Theorem 4.2. *The $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms is an Einstein manifold iff, the following conditions hold,*

$$\alpha = -2n(g_1-g_3), \quad B^a_bB^c_c = \frac{1}{2}\left[(g_1+g_2)\delta^a_b + ng_2\delta^a_b - 2n(g_1-g_3)\right].$$

Proof. Using Definition 2.3, Theorem 4.1, and the relations (7), we can derive the result. □

The following theorem establishes Einstein characteristics for $\mathcal{GS}$-space forms, generalizing Theorem 3.4 with secondary curvature β.

Theorem 4.3. *The $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms is an quasi-Einstein manifold iff, the following relations are hold,*

$$\alpha+\beta = -2n(g_1-g_3), \quad B^a_bB^c_c = \frac{1}{2}\left[(g_1+g_2)\delta^a_b + ng_2\delta^a_b - [2n(g_1-g_3)]\delta^a_b\right].$$

Proof. Employing the relations (7), Definition 2.3, and Theorem 4.1, the desired results are obtained. □

The components of pseudo quasi-conformal tensor forms for $\mathcal{GS}$-space forms are derived in the following theorem, which extends Theorem 3.1 by employing g_1, g_2 and g_3.

Theorem 4.4. *In the $\mathcal{AG}_S$-space, the components of the pseudo quasi-conformal tensor of the $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms are given as follows,*

1. $\mathcal{Q}_{\hat{a}bc\hat{d}} = (\mathrm{p}+\mathrm{r})[(g_1+g_2)\delta^a_c\delta^d_b + 2g_2\delta^a_b\delta^d_c] + \left(\mathrm{q}-\frac{\mathrm{r}}{2n}\right)\mathcal{S}^d_b\delta^a_c + \mathrm{q}\mathcal{S}^a_c\delta^d_b - \frac{\mathcal{K}\delta^d_b\delta^a_c}{2n(2n+1)}(\mathrm{p}+4n\mathrm{q}),$

2. $\mathcal{Q}_{\hat{a}0b0} = (\mathrm{p}+\mathrm{r})(g_1 - g_3)\delta^a_b + \left(\mathrm{q} - \frac{\mathrm{r}}{2n}\right)\mathcal{S}_{00}\delta^a_b + \mathrm{q}\mathcal{S}^a_b g_{00} - \frac{\mathcal{K}\delta^a_b}{2n(2n+1)}(\mathrm{p}+4n\mathrm{q})$,

3. $\mathcal{Q}_{\hat{a}\hat{b}cd} = 2(\mathrm{p}+\mathrm{r})[(g_1-g_2)\delta^a_c\delta^d_b + (g_2+g_1)\delta^a_d\delta^b_c] + \left(\frac{\mathrm{r}}{n}\right)\mathcal{S}^{[a}_d\delta^{b]}_c + 4\mathrm{q}\mathcal{S}^{[a}_{[c}\delta^{b]}_{d]} - \frac{\mathcal{K}\delta^{ab}_{cd}}{2n(2n+1)}(\mathrm{p}+4n\mathrm{q})$.

Proof. According to the Theorems 3.1 and 3.11 and the relations (7), the required results are obtained. □

Definition 4.1. *A $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms is called a ξ-pseudo quasi-conformally flat (ξ-$\mathcal{PQC}$-flat), if,*

$$\mathcal{G}(\mathcal{Q}(W,P)\xi, Y) = 0. \tag{8}$$

The subsequent theorem demonstrates that ξ-$\mathcal{PQC}$-flat of $\mathcal{GS}$-space forms is quasi-Einstein, supporting Theorem 3.13.

Theorem 4.5. *A ξ-$\mathcal{PQC}$-flat $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms is quasi-Einstein Manifold with,*

$$\alpha = \frac{1}{\mathrm{q}}\left[(2n\mathrm{q} - 2\mathrm{r} - \mathrm{p})(g_1 - g_3) + \frac{\mathcal{K}\delta^a_b}{2n(2n+1)}(\mathrm{p}+4n\mathrm{q})\right], \quad \beta = \mathcal{S}_{00} - \alpha.$$

Proof. Assume that $\mathcal{M}$ is a ξ-$\mathcal{PQC}$-flat $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms. Using (8), we have,

$$\mathcal{G}(\mathcal{Q}(W,P)\xi, Y) = 0.$$

In the $\mathcal{AG}_S$-space, the above equality can be written as,

$$\mathcal{Q}_{\hat{a}0b0} = \mathcal{Q}_{\hat{c}0\hat{a}b} = \mathcal{Q}_{c0a\hat{b}} = 0.$$

Based on $\mathcal{Q}_{\hat{a}0b0} = 0$, we conclude that,

$$(\mathrm{p}+\mathrm{r})(g_1 - g_3)\delta^a_b + \left(\mathrm{q} - \frac{\mathrm{r}}{2n}\right)\mathcal{S}_{00}\delta^a_b + \mathrm{q}\mathcal{S}^a_b g_{00} - \frac{\mathcal{K}\delta^a_b}{2n(2n+1)}(\mathrm{p}+4n\mathrm{q}) = 0.$$

This implies,

$$\mathcal{S}^a_b = \frac{1}{\mathrm{q}}\left[(2n\mathrm{q} - 2\mathrm{r} - \mathrm{p})(g_1 - g_3) + \frac{\mathcal{K}\delta^a_b}{2n(2n+1)}(\mathrm{p}+4n\mathrm{q})\right]\delta^a_b.$$

Thus, $\mathcal{M}$ is a quasi-Einstein manifold with,

$$\alpha = \frac{1}{\mathrm{q}}\left[(2n\mathrm{q} - 2\mathrm{r} - \mathrm{p})(g_1 - g_3) + \frac{\mathcal{K}\delta^a_b}{2n(2n+1)}(\mathrm{p}+4n\mathrm{q})\right], \quad \beta = \mathcal{S}_{00} - \alpha.$$

□

The forthcoming theorem computes scalar curvature for ξ-$\mathcal{PQC}$-flat $\mathcal{GS}$-space forms under Einstein conditions, building on Theorem 3.5.

Theorem 4.6. *Let $\mathcal{M}$ be an Einstein manifold with a ξ-$\mathcal{PQC}$-flat $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms, then the scalar curvature takes the formula,*

$$\mathcal{K} = \frac{2n(2n+1)}{\mathrm{p}+4n\mathrm{q}}\left[-(2n\mathrm{q} - 2\mathrm{r} - \mathrm{p}) - \frac{2n}{\mathrm{q}}\right](g_1 - g_3).$$

Proof. Since $\mathcal{M}$ is an Einstein manifold, from Theorem 4.5 and $\beta = 0$, we have,

$$\mathcal{S}_{00} = \alpha = \frac{1}{\mathrm{q}}\left[(2n\mathrm{q} - 2\mathrm{r} - \mathrm{p})(g_1 - g_3) + \frac{\mathcal{K}\delta_b^a}{2n(2n+1)}(\mathrm{p} + 4n\mathrm{q})\right],$$

Therefore,

$$\mathcal{K} = \frac{2n(2n+1)}{\mathrm{p} + 4n\mathrm{q}}\left[-(2n\mathrm{q} - 2\mathrm{r} - \mathrm{p}) - \frac{2n}{\mathrm{q}}\right](g_1 - g_3).$$

□

Definition 4.2. *A $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms is said to be Φ-pseudo quasi-conformally flat(Φ-$\mathcal{PQC}$-flat), if,*

$$\mathcal{Q}(\Phi X, \Phi W, \Phi P, \Phi Y) = 0. \tag{9}$$

The following theorem demonstrates quasi-Einstein characteristics for $\mathcal{GS}$-space forms with Φ-$\mathcal{PQC}$-flat, expanding Theorem 3.12.

Theorem 4.7. *A Φ-$\mathcal{PQC}$-flat $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms is quasi-Einstein manifold with,*

$$\alpha = \frac{1}{n\mathrm{q}}\left\{\frac{(\mathrm{p}+4n\mathrm{q})\mathcal{K}}{2(2n+1)} - (\mathrm{p}+\mathrm{r})[n(g_1+g_2)+2g_2] - \left(\mathrm{q} - \frac{\mathrm{r}}{2n}\right)\mathcal{S}_b^b\right\}, \quad \beta = \mathcal{S}_{00} - \alpha.$$

Proof. Let $\mathcal{M}$ be a Φ-$\mathcal{PQC}$-flat $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms. From (9), in the $\mathcal{AG}_S$-space, and using Theorem 4.4, we have,

$$\mathcal{Q}_{\hat{a}bcd} = \mathcal{Q}_{\hat{a}bc\hat{d}} = \mathcal{Q}_{\hat{a}b\hat{c}\hat{d}} = \mathcal{Q}_{\hat{a}\hat{b}cd} = 0.$$

From $\mathcal{Q}_{\hat{a}bc\hat{d}} = 0$ and Theorem 3.1, we deduce,

$$(\mathrm{p}+\mathrm{r})[(g_1+g_2)\delta_c^a\delta_b^d + 2g_2\delta_b^a\delta_c^d] + \left(\mathrm{q} - \frac{\mathrm{r}}{2n}\right)\mathcal{S}_b^d\delta_c^a + \mathrm{q}\mathcal{S}_c^a\delta_b^d - \frac{\mathcal{K}\delta_b^d\delta_c^a}{2n(2n+1)}(\mathrm{p}+4n\mathrm{q}) = 0.$$

Retracting the above relation via the indices (d, b), implies,

$$(\mathrm{p}+\mathrm{r})[n(g_1+g_2)\delta_c^a + 2g_2\delta_c^a] + \left(\mathrm{q} - \frac{\mathrm{r}}{2n}\right)\mathcal{S}_b^b\delta_c^a + n\mathrm{q}\mathcal{S}_c^a - \frac{\mathcal{K}\delta_c^a}{2(2n+1)}(\mathrm{p}+4n\mathrm{q}) = 0.$$

Hence,

$$\mathcal{S}_c^a = \alpha\delta_c^a.$$

Therefore, $\mathcal{M}$ is a quasi-Einstein manifold with,

$$\alpha = \frac{1}{n\mathrm{q}}\left\{\frac{(\mathrm{p}+4n\mathrm{q})\mathcal{K}}{2(2n+1)} - (\mathrm{p}+\mathrm{r})[n(g_1+g_2)+2g_2] - \left(\mathrm{q} - \frac{\mathrm{r}}{2n}\right)\mathcal{S}_b^b\right\}, \quad \beta = \mathcal{S}_{00} - \alpha.$$

□

Based on Theorem 3.13, the next theorem provides scalar curvature formula for Einstein $\mathcal{GS}$-space forms with Φ-$\mathcal{PQC}$-flat conditions.

Theorem 4.8. *Suppose that $\mathcal{M}$ is an Einstein manifold with Φ-$\mathcal{PQC}$-flat $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms, then the scalar curvature is given by,*

$$\mathcal{K} = \frac{2(2n+1)}{\mathrm{p}+4\mathrm{q}n}\left\{(\mathrm{p}+\mathrm{r})[n(g_1+g_2)+2g_2]+\left(\mathrm{q}-\frac{\mathrm{r}}{2n}\right)\mathcal{S}_b^b-2n^2\mathrm{q}(g_1-g_3)\right\}.$$

Proof. Since $\mathcal{M}$ is an Einstein manifold, we obtain the following by applying Theorem 4.7 and taking $\beta=0$,

$$\mathcal{S}_{00}=\alpha=\frac{1}{n\mathrm{q}}\left\{\frac{(\mathrm{p}+4n\mathrm{q})\mathcal{K}}{2(2n+1)}-(\mathrm{p}+\mathrm{r})[n(g_1+g_2)+2g_2]-\left(\mathrm{q}-\frac{\mathrm{r}}{2n}\right)\mathcal{S}_b^b\right\},$$

Hence,

$$\mathcal{K} = \frac{2(2n+1)}{\mathrm{p}+4\mathrm{q}n}\left\{(\mathrm{p}+\mathrm{r})[n(g_1+g_2)+2g_2]+\left(\mathrm{q}-\frac{\mathrm{r}}{2n}\right)\mathcal{S}_b^b-2n^2\mathrm{q}(g_1-g_3)\right\}.$$

□

Definition 4.3. *A $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms is said to be a quasi- pseudo-quasi conformally flat($\mathcal{QPQ}$-flat), if,*

$$\mathcal{Q}(X,W,P,\Phi Y)=0. \tag{10}$$

In accordance with Theorem 3.13, the following theorem identifies $\mathcal{GS}$-space forms with $\mathcal{QPQ}$-flat as quasi-Einstein.

Theorem 4.9. *A $\mathcal{QPQ}$-flat $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms is quasi-Einstein Manifold with,*

$$\alpha=\frac{1}{\left(\mathrm{q}(n-2)+\frac{\mathrm{r}}{2n}\right)}\left\{\frac{(n-1)(\mathrm{p}+4n\mathrm{q})\mathcal{K}}{2n(2n+1)}-2(\mathrm{p}+\mathrm{r})[(n-1)g_1+(1-n)g_2]-\left(\mathrm{q}-\frac{\mathrm{r}}{2n}\right)\mathcal{S}_d^d\right\},$$
$$\beta=\mathcal{S}_{00}-\alpha.$$

Proof. Let $\mathcal{M}$ be a $\mathcal{QPQ}$-flat $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space form. Using relation (10) and Theorem 4.4, on the $\mathcal{AG}_S$-space, we have,

$$2(\mathrm{p}+\mathrm{r})[(g_1-g_2)\delta_c^a\delta_b^d+(g_2+g_1)\delta_d^a\delta_c^b]+\left(\frac{\mathrm{r}}{n}\right)\mathcal{S}_d^{[a}\delta_c^{b]}+4\mathrm{q}\mathcal{S}_{[c}^{[a}\delta_{d]}^{b]}-\frac{\mathcal{K}\delta_{cd}^{ab}}{2n(2n+1)}(\mathrm{p}+4n\mathrm{q})=0.$$

Retracting the last equation by the indices (d,b), we deduce,

$$2(\mathrm{p}+\mathrm{r})[n(g_1-g_2)\delta_c^a+(g_2+g_1)\delta_c^a]+\left(\mathrm{q}(n-2)+\frac{\mathrm{r}}{2n}\right)\mathcal{S}_c^a+\left(\mathrm{q}-\frac{\mathrm{r}}{2n}\right)\mathcal{S}_d^d-\frac{(n-1)\mathcal{K}\delta_c^a}{2n(2n+1)}(\mathrm{p}+4n\mathrm{q})=0.$$

Hence,

$$\mathcal{S}_c^a=\alpha\delta_c^a.$$

Therefore, $\mathcal{M}$ is quasi-Einstein with,

$$\alpha=\frac{1}{\left(\mathrm{q}(n-2)+\frac{\mathrm{r}}{2n}\right)}\left\{\frac{(n-1)(\mathrm{p}+4n\mathrm{q})\mathcal{K}}{2n(2n+1)}-2(\mathrm{p}+\mathrm{r})[(n-1)g_1+(1-n)g_2]-\left(\mathrm{q}-\frac{\mathrm{r}}{2n}\right)\mathcal{S}_d^d\right\},$$
$$\beta=\mathcal{S}_{00}-\alpha.$$

□

The concluding theorem computes scalar curvature for $\mathcal{GS}$-space forms with $\mathcal{QPQ}$-flat conditions under Einstein constraints, which improves Theorem 3.13.

Theorem 4.10. *Suppose that $\mathcal{M}$ is an Einstein manifold with $\mathcal{QSA}_S$-manifold of $\mathcal{GS}$-space forms, then the scalar curvature has the formula,*

$$\mathcal{K} = \frac{2n(2n+1)}{(n-1)(\mathrm{p}+4n\mathrm{q})}\left\{2(\mathrm{p}+\mathrm{r})[(n-1)g_1+(1-n)g_2]+\left(\mathrm{q}-\frac{\mathrm{r}}{2n}\right)\mathcal{S}_d^d-2\left(\mathrm{q}n(n-2)+\frac{\mathrm{r}}{2}\right)(g_1-g_3)\right\}.$$

Proof. Suppose that $\mathcal{M}$ is an Einstein manifold, using Theorem 4.9 and taking $\beta = 0$, we have,

$$\mathcal{S}_{00} = \alpha = \frac{1}{\left(\mathrm{q}(n-2)+\frac{\mathrm{r}}{2n}\right)}\left\{\frac{(n-1)(\mathrm{p}+4n\mathrm{q})\mathcal{K}}{2n(2n+1)} - 2(\mathrm{p}+\mathrm{r})[(n-1)g_1+(1-n)g_2]-\left(\mathrm{q}-\frac{\mathrm{r}}{2n}\right)\mathcal{S}_d^d\right\},$$

Therefore,

$$\mathcal{K} = \frac{2n(2n+1)}{(n-1)(\mathrm{p}+4n\mathrm{q})}\left\{2(\mathrm{p}+\mathrm{r})[(n-1)g_1+(1-n)g_2]+\left(\mathrm{q}-\frac{\mathrm{r}}{2n}\right)\mathcal{S}_d^d-2\left(\mathrm{q}n(n-2)+\frac{\mathrm{r}}{2}\right)(g_1-g_3)\right\}.$$

□

5 Conclusions

This study identified the components of the Ricci tensor and pseudo quasi-conformal curvature tensor for $\mathcal{QSA}_S$-manifolds of $\mathcal{GS}$-space forms. Different types of $\mathcal{QSA}_S$-manifolds were presented and investigated, and their relationships to $\mathcal{GS}$-space forms were examined. Under ξ-pseudo quasi conformal conditions, $\mathcal{QSA}_S$-manifolds demonstrate quasi Einstein properties, with specific conditions established for quasi pseudo $\mathcal{QSA}_S$-manifold. Finally, scalar curvatures were calculated under these conditions, enhancing our comprehension of these geometric structures.

Acknowledgement The authors are very grateful to the referees and editors for their useful comments and valuable suggestions that helped improve this work.

Conflicts of Interest The authors declare that there is no conflict of interest regarding the publication of this article.

References

[1] M. Y. Abass & H. M. Abood (2019). ϕ-holomorphic sectional curvature and generalized Sasakian space forms for a class of Kenmotsu type. *Journal of Basrah Researches (Sciences)*, *45*(2), 108–117.

[2] H. M. Abood & F. H. AlHussaini (2020). On the conharmonic curvature tensor of a locally conformal almost cosymplectic manifold. *Communications of the Korean Mathematical Society*, *35*(1), 269–278. https://doi.org/10.4134/CKMS.c190003.

[3] A. Abu Saleem & A. R. Rustanov (2015). Curvature identities special generalized manifolds Kenmotsu second kind. *Malaysian Journal of Mathematical Sciences*, *9*(2), 187–207.

[4] P. Alegre & A. Carriazo (2008). Structures on generalized Sasakian-space-forms. *Differential Geometry and Its Applications*, *26*(6), 656–666. https://doi.org/10.1016/j.difgeo.2008.04.014.

[5] F. H. AlHussaini, A. R. Rustanov & H. M. Abood (2020). Vanishing conharmonic tensor of normal locally conformal almost cosymplectic manifold. *Commentationes Mathematicae Universitatis Carolinae*, *61*(1), 93–104. https://doi.org/10.14712/1213-7243.2020.008.

[6] D. E. Blair (1967). The theory of quasi-Sasakian structures. *Journal of Differential Geometry*, *1*(3–4), 331–345. https://doi.org/10.4310/JDG/1214428097.

[7] A. Carriazo (2005). On generalized Sasakian-space-forms. In *Proceedings of the Ninth International Workshop on Differential Geometry*, volume 9 pp. 31–39.

[8] É. Cartan (2001). *Riemannian Geometry in an Orthogonal Frame: From Lectures Delivered by Élie Cartan at the Sorbonne in 1926–1927*. World Scientific, Singapore.

[9] M. C. Chaki & R. K. Maity (2000). On quasi Einstien manifolds. *Publicationes Mathematicae Debrecen*, *57*(3–4), 297–306. https://doi.org/10.5486/PMD.2000.2169.

[10] U. C. De, A. K. Sengupta & C. Calin (2004). Generic submanifolds of quasi-Sasakian manifolds. *Demonstratio Mathematica*, *37*(2), 429–438. https://doi.org/10.1515/dema-2004-0220.

[11] S. Kanemaki (1977). Quasi-Sasakian manifolds. *Tohoku Mathematical Journal*, *29*(2), 227–233. https://doi.org/10.2748/tmj/1178240654.

[12] S. Kanemaki (1984). On quasi-Sasakian manifolds. *Banach Center Publications*, *12*(1), 95–125.

[13] V. F. Kirichenko (1988). Methods of generalized Hermitian geometry in the theory of almost-contact manifolds. *Journal of Soviet Mathematics*, *42*, 1885–1919. https://doi.org/10.1007/bf01094419.

[14] V. F. Kirichenko (1994). Generalized quasi-Kaehlerian manifolds and axioms of CR-submanifolds in generalized Hermitian geometry, ii. *Geometriae Dedicata*, *52*(1), 53–85. https://doi.org/10.1007/BF01263524.

[15] V. F. Kirichenko & A. R. Rustanov (2002). Differential geometry of quasi-Sasakian manifolds. *Sbornik: Mathematics*, *193*(8), 1173–1201. https://doi.org/10.1070/SM2002v193n08ABEH000675.

[16] D. H. Lim (2016). On characterizations of real hypersurface related to the shape operator and structure tensor in nonflat complex space form. *Malaysian Journal of Mathematical Sciences*, *10*(3), 245–252.

[17] A. K. Mondal, A. De et al. (2011). Some theorems on 3-dimensional quasi-Sasakian manifolds. *Tamsui Oxford Journal of Information and Mathematical Sciences*, *27*(4), 411–427.

[18] Z. Olszak (1996). On three-dimensional conformally flat quasi-Sasakian manifolds. *Periodica Mathematica Hungarica*, *33*(2), 105–113. https://doi.org/10.1007/BF02093508.

[19] S. Y. Perktaş & A. Yıldız (2020). On quasi-Sasakian 3-manifolds with respect to the Schouten-van Kampen connection. *International Electronic Journal of Geometry*, *13*(2), 62–74. https://doi.org/10.36890/iejg.742073.

[20] S. Rahman (2022). Some results on the geometry of warped product CR-submanifolds in quasi-Sasakian manifold. *Cubo (Temuco)*, *24*(1), 105–114. https://doi.org/10.4067/S0719-06462022000100105.

[21] S. Sasaki & Y. Hatakeyama (1961). On differentiable manifolds with certain structures which are closely related to almost contact structure. ii. *Tohoku Mathematical Journal*, *13*(2), 281–294. https://doi.org/10.2748/tmj/1178244304.

[22] A. A. Shaikh (2005). A pseudo quasi-conformal curvature tensor on a Riemannian manifold. *South East Asian Journal of Mathematics and Mathematical Sciences*, *14*, 15–20.

[23] S. Tanno (1971). Quasi-Sasakian structures of rank $2p+1$. *Journal of Differential Geometry*, *5*(3-4), 317–324. https://doi.org/10.4310/JDG/1214429995.

[24] V. Venkatesha & B. Shanmukha (2018). W_2-curvature tensor on generalized Sasakian space forms. *Cubo (Temuco)*, *20*(1), 17–29. http://dx.doi.org/10.4067/S0719-06462018000100017.

[25] V. Venkatesha, B. Sumangala & C. S. Bagewadi (2012). On ϕ-recurrent generalized Sasakian-space-forms. *Global Journal of Science Frontier Research Mathematics and Decision Sciences*, *12*(F9), 63–70.

[26] K. R. Vidyavathi, A. Sushilabai & C. S. Bagewadi (2023). Semi-symmetric generalized Sasakian-space-forms. *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*, *34*(1), 112–119.